Why logic works

Sarvadaman Chowla memorial lecture

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2. What does logic do?

- Logic has to do with **arguments**, which are the
  'If ... then ... therefore ...' kind of stuff.

- Arguments help us arrive at **truth**.

- An argument has a **hypothesis**, and from the argument we **deduce** (or **infer**) a **conclusion** (or **inference**).

- If the hypothesis of the argument is true, and if the argument is **logically valid**, then the conclusion of the argument is true.

- This is the job of logic : it guarantees the truth of any inference made from any true hypothesis, provided the argument is logically valid.

- It has been fairly successful in this job. Our everyday life depends on it. Engineers, doctors, lawyers rely on it for life-and-death decisions. All academic subjects, especially, the edifice of mathematics, is a testimony that logic works.

3. Questions

Now that we have noted what logic does for us, many questions naturally arise. To begin with, we ask:

- **Why** does logic work?

- **How** does it work?

- What are the **conditions** that it needs in order to work?

- In what **domain**, or subject matter, are these conditions fulfilled?

- What are the **limitations of logic**, if any, even where it works well such as in mathematics?

- What are the **alternatives** to standard logic?

- When are the conditions needed by logic not fulfilled? How can we understand such things?

4. **Sample logical and illogical arguments**

- All men are mortal. Socrates is a man. Therefore, Socrates is mortal.
  (Basic example of a logical argument by Aristotle, called 'logical syllogism'.)

- All eagles can fly high. This bird is flying high. Therefore, this bird must be an eagle.
  (Illogical! The conclusion can be wrong as there exist other birds that can fly high.)

- Some Indians can speak Hindi. Some Indians can speak Tamil. Therefore, some Indians can speak both Hindi and Tamil.
  (Illogical, even though the conclusion is correct!)

5. **The languages logic**

- **Natural languages**, such as Marathi, Sanskrit, Hindi, English etc. are very complicated.

- Lot of logical arguments can be carried out in a **simplified language**.

- A **first order language** is a kind of simplified language, with precise **vocabulary** and **grammar**.

- The sentences are **machine checkable in bounded number of steps** for grammatical correctness.

- The checking involves only **for** loops, and does not need **until** loops, in computer programming terms.

6. **First order languages -1**

- Such a language has some **names** (e.g. Socrates), and an infinite supply of **pronouns** x, y, z, x', x'', etc. The pronouns are called **variables**.

- It has some **predicates** applicable to one or more names or variables. For example, '... is mortal', '... is the mother of ...'. A predicate of more than one variables is also called a **relation**.

- There is always a binary relation '$\ldots = \ldots$' called equality, which **means** identity.

- There can be some **functions** of one or more variables. For example, 'the oldest child of $\ldots$ and $\ldots$'.

- A **simple sentence** is formed by inserting names or variables or the outputs of functions in all the available slot of a predicate. For example, 'x is mortal', 'x's mother = the oldest child of y and z', '$x^2 + y^2 = z^2$'.

- A **compound sentence** is formed from simple sentences through **logical connectives** and **quantifiers**, and using brackets.

7. **First order languages -2**

The logical connectives are

- **negation** $\neg$, stands for 'not'

- **conjunction** $\wedge$, stands for 'and'

- **disjunction** $\vee$, stands for 'or'

- **conditional** $\Rightarrow$, stands for 'if $\ldots$ then $\ldots$'

- **biconditional** $\Leftrightarrow$, stands for '$\ldots$ if and only if $\ldots$'

- $\neg$ and $\Rightarrow$ suffice: other connectives can be defined in their terms.

The quantifiers are

- the **universal quantifier** $\forall$, stands for 'for all'.

- the **existential quantifier** $\exists$, stands for 'for some'. It can be defined in terms of $\forall$ and $\neg$.

8. **First order languages -3**

We illustrate this with the language L_A of arithmetic and the language L_S of set theory.

- L_A has 0 and 1 as constants. L_S has $\emptyset$ as a constant (the symbol for empty set).

- Symbols for variables are x, y, z, x', x'' etc. These are supposed to range over natural numbers $0, 1, 2, \ldots$ for L_A, and over all sets for L_S.

- L_A has the successor function Sx, that is, Sx denotes the successor of x, and functions and $x + y$ and $x \times y$ for the addition and multiplication in L_A.

- L_S has the membership relation $x \in y$ (means x is a member of y).

- Both languages have the binary relation $=$ ('equal to').

9. **First order languages -4**

- $(\exists y)(x = y + y)$ (this is a translation in L_A of 'x is even'). This is an **open sentence** as x is **free** in it.

- $(\forall x)(\forall y)(x+y = y+x)$ (this is a translation in L_A of 'addition is commutative'). This is a **closed sentence**, as no variables are free in it.

- $\neg(\exists x)(x \in \emptyset)$ (this is a translation in L_S of 'no set is a member of the empty set'). This is a closed sentence.

- $(\forall z)(z \in x \Leftrightarrow y \in x) \Rightarrow x = y$ (this is a translation in L_S of 'if two sets x and y have the same elements then they are one and the same set'). This is an open sentence with x and y the free variables in it.

- $(\exists z)(x + Sz = y)$ (this is a translation in L_A of '$x < y$'). It has x and y as free variables. Note that Sz is always positive.

- Exercise: Translate into L_A the sentence 'x is a prime'.

9. **Truth**

- Language refers to things which are outside the language.

- Truth is a property of a sentence, which depends on what the sentence says about the external reality.

- The truth of an individual sentence, which does not use the word 'true' or its synonyms or antonyms, can be defined in a common-sense way (Tarski, 1930's) as in the following examples.

- The sentence '$2 + 2 = 4$' is true if and only if $2 + 2 = 4$.

- The sentence 'Chennai is in Panjab' is true if and only if Chennai is in Panjab. And so on.

- The notion of truth is explicitly needed when we want to generalize over sentences.

- For any closed sentence A, the sentence $A \lor \neg A$ is true.

- Baba vakyam pramanam. (Whatever Baba says is true.)

11. **Logical truth**

- **Definition 1.** A closed sentence in a first order language is **logically true** if each sentence obtained from it by replacing each simple predicate by an arbitrary new (simple or compound) predicate is again true.

- **Definition 2.** A closed sentence in a first order language is **logically true** if each of its **interpretations** is true.

- Here, an interpretation of a first order language consists of
  a set D, whose members are possible values for the variables in L,
  chosen elements of D for the names in L,
  chosen functions $D \to D$ or $D \times \ldots \times D \to D$ for the functions in L
  chosen subsets of $D \times D$, $D \times \ldots \times D$ for the relations in D.

- The equality sign $=$, the logical connectives $\neg$, $\land$, $\lor$, $\Rightarrow$, $\Leftrightarrow$, the quantifiers $\forall$, $\exists$ have their standard interpretations indicated by the terminology, and brackets have their usual role as separators.

- Extension of the concept of logical truth to open sentences.

11. **Examples of logical truths**

- $((A \Rightarrow B) \land (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$
  where A, B and C stand for closed sentences
  is a logical truth.

- $\neg(\forall x)P(x) \Rightarrow (\exists x)\neg P(x)$
  where P stands for a predicate of one variable
  is a logical truth.

- How about
  $((\exists x)P(x) \land (\exists x)Q(x)) \Rightarrow (\exists x)(P(x) \land Q(x))$
  where P and Q stand for predicates of one variable?
  Not a logical truth.

- How about
  $(\forall x)P(x) \Rightarrow P(y)$
  where P stands for a predicate of one variable?

  A logical truth.

12. Logical consequence, Examples

- A sentence A in L is a **logical consequence** of set of sentences $\{B_1, B_2, \ldots\}$ in L (called the set of **hypotheses**) if under any interpretation of the language that makes each B_i true also makes A true.

- Equivalently, if any substitution of simple sentences that makes each B_i true also makes A true, then A is a logical consequence of $\{B_1, B_2, \ldots\}$.

- A logical truth A is just the logical consequence of the set of hypotheses which is an empty set.

Some examples: we use the notation $\{B_1, B_2, \ldots\} \vDash A$ for logical consequence, and $\vDash A$ for logical truth.

- $\vDash (A \wedge B) \Rightarrow A$

- $\vDash A \vee \neg A$ (law of excluded middle)

- $\{A, A \Rightarrow B\} \vDash B$ (**modus ponens**)

- $\{(\forall x)(P(x) \Rightarrow Q(x)), P(c)\} \vDash Q(c)$

- $P(x) \vDash (\forall x)P(x)$ (this is called **generalization**)

13. Is logical truth decidable?

Logical truth just depends on the form, that is, the grammatical structure of the sentence, not on the meanings of its simple constituent predicates.

This raises the question:

- Is there a mechanical procedure to check whether a sentence is a logical truth? If the answer is yes, we would say that logical truth is **decidable**.

- More generally, given a mechanical way of generating a set of hypothesis $B_1, B_2, \ldots$ as the input, is there a mechanical way to check whether a sentence A is their logical consequence?

Theorem (Church and Turing, 1930's) Logical truth is undecidable.

14 Logical proofs

Valid arguments establish the truth of the conclusion if the assumptions are true. This leads to the notion of a formal proof.

We first identify a small number of formats ('schemas') of logical truths, which we call **logical axioms**, and just two formats of valid deduction **MP** (modus ponens) and **Gen** (generalization).

Given some hypothesis $\{P_1, P_2, \ldots\}$, if we have a sequence of sentences $A_1, \ldots A_n$ such that for each i,

A_i a logical axiom, or

A_i is one of the P_j's, or

A_i is obtained from the previous statements $A_1, \ldots, A_{i-1}$ by an application of MP or Gen,

then we say that the sequence $A_1, \ldots A_n$ is a **formal proof** of A_n from the hypothesis $P_1, P_2, \ldots$. Symbolically, we write
$$\{P_1, P_2, \ldots\} \vdash A_n$$

Clearly, if $\{P_1, P_2, \ldots\} \vdash A_n$ then $\{P_1, P_2, \ldots\} \vDash A_n$.

Is the converse true?

15 List of all axioms and deduction rules

- Logical axioms: Three propositional axiom schema.
  (1) $A \Rightarrow (B \Rightarrow A)$
  (2) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)$
  (3) $(\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)$

- Three quantifier axiom schema.
  (4) $(\forall x_i)A \Rightarrow A$ if x_i does not occur free in A.
  (5) $(\forall x_i)A \Rightarrow A(x_i/t)$ whenever the variable x_i is free in A, and t is any term which is *free for x_i in A*.
  (6) $(\forall x_i)(A \Rightarrow B) \Rightarrow (A \Rightarrow (\forall x_i)B)$ if x_i does not occur free in A.

- Two deduction rules.
  MP: From A and $A \Rightarrow B$ we can deduce B.
  Gen: From A we can deduce $(\forall x_i)A$.

- Three axiom (schema) of equality. (E1) $(\forall x)(x = x)$.
  (E2) Replacing a term by an equal term inside a function gives equal values.
  (E3) Replacing a term by an equal term inside a relation gives a new statement which is implied by the old statement.

16 Godel's completeness theorem

- We saw that if $\{P_1, P_2, \ldots\} \vdash A_n$ then $\{P_1, P_2, \ldots\} \vDash A_n$, that is, proofs establish logical consequences.

- But can proofs capture all possible logical consequences?

- The answer is 'yes'. This was proved by Gödel in his PhD thesis in 1920's, when he was a young graduate student.

- So with this, we can now say that $\{P_1, P_2, \ldots\} \vdash A_n$ if and only if $\{P_1, P_2, \ldots\} \vDash A_n$.

- In particular, a statement A is a logical truth if and only if A can be logically proved from the axioms, using the deduction rules.

- Consequence: the set of logical truths in a language L is mechanically enumerable: we can program a computer to print a list of all of them (though this will go on and on).

- This does not contradict the undecidability of logical truth!

17 Formal theory

We require a formal theory T to have the following features.

- There should be a first order language L for the theory.

- The standard axioms of first order logic, and the deduction rules MP and Gen should be assumed.

- There should be a set of axioms for the theory (over and above the logical axioms).

- Whether a sentence A in L is an axiom should be mechanically checkable in bounded number of steps.

A **model** for the theory consists of an interpretation of the language L (means a domain D for the names and variables, and subsets of D, $D \times D$, etc. for the predicates of L) such that all the axioms of T become true statements.

Example: The theory PA (Peano Arithmetic) has L_A as its language, the Peano Axioms as its axioms, and $D = \{0, 1, 2, 3, \ldots\}$ with the usual interpretations for $=$, 0, 1, S, $+$, $\times$.

18 Peano axioms

- Robinson axioms.
  (1) $(\forall x)(Sx \neq 0)$
  (2) $(\forall x)(\forall y)(Sx = Sy \Rightarrow x = y)$
  (3) $(\forall x)((x \neq 0) \Rightarrow (\exists y)(Sy = x))$
  (4) $(\forall x)(x + 0 = x)$
  (5) $(\forall x)(\forall y)(x + Sy = S(x + y))$
  (6) $(\forall x)(x \times 0 = 0)$
  (7) $(\forall x)(\forall y)(x \times Sy = x \times y + x)$

- Induction axiom schema. For each A we have an axiom:

$$(A(0) \wedge (\forall x)(A(x) \Rightarrow A(Sx))) \Rightarrow (\forall x)A(x).$$

  If A has other free variables besides x, then universally quantify the above formula over them.

19 Arithmetization of syntax

- In any formal language, we can attach a unique number to each grammatically correct term or sentence, and to each finite sequence of sentences. This is called Gödel numbering. The coding or decoding between expressions and their Gödel numbers can be done mechanically done in bounded number of steps.

- The rules of grammar and of logical deduction become arithmetical relations between the numbers.

- If the language L has a (defined) predicate which says 'x is a natural number', and there are the standard arithmetical symbols $=$, 0, 1, S, $+$, $\times$ available in L, then there is a purely arithmetical predicate $Thm(n)$ expressible in L, which says that n is a natural number which is the Gödel number of a formal theorem in the theory T.

- A formal theory T is consistent if there is no sentence A such that both A and $\neg A$ are theorems of T. Equivalently, $\neg(0 = 0)$ should not be a theorem of T. If g denotes the Gödel number of $\neg(0 = 0)$, this means that T is consistent if and only if $\neg Thm(g)$ is a true statement of arithmetic.

20 Godel's incompleteness theorem

The hope that we can set up a formal theory for arithmetics (or more generally, for all of mathematics) such that all truths will become theorems is dashed by the following famous result.

- **Theorem** (Gödel, 1930) Let T be a formal theory whose axioms are true and whose language L can express basic arithmetic. Then the arithmetical sentence $\neg Thm(g)$ in L, which expresses 'T is consistent' in the language L, is a true sentence cannot be proved in T.

- The above theorem can be applied even to an enhanced theory which has the truth $\neg Thm(g)$ as an extra axiom, and so on!

- Thus, truth cannot be captured via theoremhood for any theory which has a basic amount of arithmetic included in it.

- Syntactic version of the above does not need the notion of truth.

- Tarski's theorem on formal undefinability of truth within L.

21 When does first order logic work?

- Robust individuals.

- A few prescribed predicates which apply to these. The robustness of their truth value.

- Everything we want to say can be said by the resulting sentences of a formal language.

- The 'standard meanings' of logical operations apply.

Then we can safely trust the deductions made using logic from hypothesis which are true.

The real world, where we use natural languages, seems to be much more complicated!

22 Natural language: some examples

- This sentence is false. (Liar paradox.)

- Is 'non-self-describing' a non-self-describing phrase?

- Is the set of all sets which are not members of themselves a member of itself? (Russell's paradox.)

- Some girls in my college only talk among themselves. (This sentence is quite unlike the sentence "Some girls in my college only speak English", for it needs sets.)

- The phrase "the smallest natural number that is not definable using less than twenty words" defines it in less than twenty words.

- A man can never truly say that he never talks about himself.

- There have been heroic attempts on the part of philosophers – both western and Indian – to make sense of common language and arguments made in it.

- Indian logic. Navyanyaya.

23 Language and the limitations of logic

- Natural language has unclear boundaries.

- Thought seems to exist beyond language.

- Infants. Animals. Rapid thinking. Music.

- Our minds are products of evolution. They are not formal systems.

- The domain of precise language, logic, and language based rationality is just the middle domain of our experience of the world.

avyaktaadeeni bhootaani vyaktamadhyaani bhaarata
avyaktanidhannyeva tatra kaa paridevanaa -(Bhagvadgita, 2.28)

'The origin of beings is inexpressible, the middle is expressible,
the end is again inexpressible. What is there to lament about it?'

There is life before and beyond logic –
we necessarily go on as integral parts of the universe!